

# The multitrace matrix model: An alternative to Connes NCG and IKKT model

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We present a new multitrace matrix model, which is a generalization of the real quartic one matrix model, exhibiting dynamical emergence of a fuzzy two-sphere and its non-commutative gauge theory. This provides a novel and a much simpler alternative to Connes non-commutative geometry and to the IKKT matrix model for emergent geometry in two dimensions.

The real quartic multitrace model [1]

$$V_0 = B \text{Tr} M^2 + C \text{Tr} M^4, \quad (1)$$

has two stable phases: *i*) the disordered (symmetric one-cut) phase and *ii*) the non-uniform ordered (two-cut) phase with the transition being identified of third order. The uniform ordered (Ising asymmetric one-cut) phase is metastable in this theory [2]. It was discovered in [23, 24] that the Ising phase becomes stable if we add to  $V_0$ , with a particular choice of the parameters  $D, B', C', D', A', \dots$ , the quartic multitrace model

$$V_1 = D(\text{Tr} M^2)^2 + B'(\text{Tr} M)^2 + C' \text{Tr} M \text{Tr} M^3 + D'(\text{Tr} M)^4 + A' \text{Tr} M^2(\text{Tr} M)^2 + \dots \quad (2)$$

These are the most general terms consistent with the multitrace expansion of the kinetic term of non-commutative  $\Phi^4$  theory at quartic order. For example, the coefficients on the fuzzy sphere [6, 7] were calculated in [3, 4] and on the fuzzy disc [8] were calculated in [5]. Another set of parameters which give a stable Ising phase are given by [23]

$$D = \frac{3N}{4}, \quad B' = \frac{\sqrt{N}}{2}, \quad C' = -N, \quad D' = 0, \quad A' = 0. \quad (3)$$

This can be explicitly verified by computing the critical exponents across the Ising transition line and the Wigner semi-circle law near in the perturbative regime [23]. The critical exponents are found to be consistent with the Onsager values [9] suggesting that the dimension of the underlying space is two and that the above matrix model in this phase is in the Ising universality class. The Wigner semi-circle law is determined by the free propagator and thus by the metric aspects of the underlying space [15–18]. The operator  $\text{Tr} M \text{Tr} M^3$  is the crucial ingredient in stabilizing the Ising phase. Indeed, given hindsight, a simpler set of parameters is given simply by  $C' = -N$  while setting all the other parameters to zero.

The configuration  $M = a \mathbf{1}_{2N}$ , for our case (3), is a solution of the equation of motion iff

$$a = 0(\text{disordered}),$$

$$a^2 = -\frac{\tilde{B} + 1}{\sqrt{N}(2\tilde{C} - 1)}(\text{uniform ordered}). \quad (4)$$

The large  $N$  scaling of the various coupling constants is given by

$$\tilde{B} = \frac{B}{N^{3/2}}, \quad \tilde{C} = \frac{C}{N^2}. \quad (5)$$

Stability requires that  $\tilde{B} \leq -1$  and  $\tilde{C} \geq 1/2$ . The configuration  $M = a\gamma$  where  $\gamma^2 = 1$  with  $\gamma$  containing an equal numbers of  $+1$  and  $-1$  is also a solution iff

$$a^2 = -\frac{\tilde{B}}{\sqrt{N}(2\tilde{C} + 3)}(\text{non uniform ordered}). \quad (6)$$

By substituting the configurations  $M = a \mathbf{1}_{2N}$  and  $M = a\gamma$  in the Schwinger-Dyson identity (with  $V = V_0 + V_1$ )

$$\frac{\langle V \rangle}{N^2} = \frac{1}{4} + \frac{B}{2} \frac{\langle \text{Tr} M^2 \rangle}{N^2} + \frac{B'}{2} \frac{\langle (\text{Tr} M)^2 \rangle}{N^2}, \quad (7)$$

we obtain the energies  $E_1$  and  $E_2$  in the two phases. It is not difficult to show that  $E_1 \leq E_2$  iff (recall that  $\tilde{B} + 1 \leq 0, 2\tilde{C} - 1 \geq 0$ )

$$\tilde{B} + 1 \leq -\frac{2\tilde{C} - 1}{4}. \quad (8)$$

This result is confirmed non-perturbatively using Monte Carlo simulation of the eigenvalue problem corresponding to the diagonalization of the quartic multitrace matrix model  $V = V_0 + V_1$ . Thus, there exists a region in the phase diagram in which the uniform ordered configuration is more stable than the disordered and the non-uniform ordered configurations [23, 24].

The partition function of the theory is given by

$$Z = \int \mathcal{D}M \exp(-V[M]). \quad (9)$$

We will assume now that the matrix  $M$  is  $2N \times 2N$ . Then, without any loss of generality, we can expand the matrix  $M$  as

$$M = M_0 \mathbf{1}_{2N} + M_1, \quad \text{Tr} M_1 = 0. \quad (10)$$

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Hence

$$M_1 = \sigma_a X_a, \quad M_0 = a + m, \quad (11)$$

where  $\sigma_a$  are the standard Pauli matrices,  $m$  is the fluctuation in the zero mode, and  $X_a$  are three hermitian  $N \times N$  matrices. By substitution, we obtain immediately the model

$$Z = \int \mathcal{D}X_a \exp(-V[X]) \int dm \exp(-f[m]). \quad (12)$$

The potential  $V$  is given now by the  $SO(3)$ -symmetric three matrix model

$$V = -C \text{Tr}[X_a, X_b]^2 + 2C \text{Tr}(X_a^2)^2 + 4D(\text{Tr}X_a^2)^2 + 2(B + \beta_0 a^2) \text{Tr}X_a^2 + 2ia\gamma\epsilon_{abc} \text{Tr}X_a X_b X_c. \quad (13)$$

The new coefficients  $\beta_0$  and  $\gamma$  are given below.

We observe that the Chern-Simons term is proportional to the value  $a$  of the order parameter. Thus, it is non-zero only in the Ising phase, and as a consequence, by tuning the parameters appropriately to the region in the phase diagram where the Ising phase exists, we will induce a non-zero value for the Chern-Simons. This is effectively the Myers term responsible for the condensation of the geometry [10, 11]. The above multitrace three matrix model is then precisely a random matrix theory describing non-commutative gauge theory on the fuzzy sphere, where the first term is the Yang-Mills piece, whereas the second and fourth terms combine to give mass and linear terms for the normal scalar field on the sphere (recall that  $a$  runs from 1 to 3). The third doubletrace term, proportional to  $D$ , depends only on the zero mode of the normal scalar field. Thus, it is not expected to play a major role in our discussion here. If we simply set  $D = 0$  then we get essentially the random matrix theory describing non-commutative gauge theory on the fuzzy sphere found in [19] (notice that the parameter  $C$  appears in front of the Yang-Mills as well as in front of the normal mass term and thus can be scaled away). This theory itself is a generalization of the stringy non-commutative gauge theory on the fuzzy sphere considered in [12–14].

Furthermore, we have shown recently in [22] that the above model with  $D = 0$  sustains an absolutely stable emergent fuzzy two-sphere geometry, and as a consequence, the expansion around this emergent geometry to obtain a non-commutative gauge theory is fully consistent for all values of the gauge coupling constant. In other words, there is no phase transition to a Yang-Mills matrix phase at a finite value of the gauge coupling constant. This is expected to hold also for  $D \neq 0$ .

However, we should emphasize here that we have obtained dynamically this slightly generalized version of non-commutative gauge theory on the fuzzy sphere by going to the phase of the model where a non-zero uniform order persists, and then by expanding around this

order, we secure a non-zero Chern-Simons term crucial for the underlying emergent geometry of the fuzzy sphere. Recall that in [19], this was achieved by constraining the matrix  $M$  directly in a particular way.

What is the effect of the zero mode  $m$ ?

The potential  $f$  in (12) of the zero mode  $m$  is given by

$$f = [2i\gamma\epsilon_{abc} \text{Tr}X_a X_b X_c + 4\alpha a^3 + 4\beta_0 a \text{Tr}X_a^2 + 2a\beta_1]m + [2\beta_0 \text{Tr}X_a^2 + \beta_1 + 6\alpha a^2]m^2 + 4\alpha a m^3 + \alpha m^4. \quad (14)$$

The various new coefficients, for our case (3), are given by

$$\begin{aligned} \beta_0 &= 3N^2(2\tilde{C} - 1), \quad \beta_1 = 2N^{5/2}(\tilde{B} + 1) \\ \alpha &= N^3(2\tilde{C} - 1), \quad \gamma = 2N^2(2\tilde{C} - 1). \end{aligned} \quad (15)$$

The integration over  $m$  can be done. The leading contribution in the large  $N$  limit is essentially the one-loop result and it is by construction subleading compared to (13). This integral consists of some function of  $\text{Tr}X_a^2$  and  $i\epsilon_{abc} \text{Tr}X_a X_b X_c$ . The conclusion of the foregoing discussion remains practically unchanged with the addition of these multitrace corrections.

Thus, the multitrace one matrix model (2), which involves only a single hermitian matrix  $M$  with  $U(2N)$  symmetry, and which exhibits a uniform order for some values of the mass parameter  $B$  and the quartic coupling constant  $C$ , provides a new mechanism for emergent fuzzy two-sphere and its non-commutative gauge theory. Indeed, by expanding around the uniform order, and then performing the integral over the zero mode  $m$ , it is seen that the resulting theory given by the mostly (since  $D \neq 0$ ) single trace  $SO(3)$ -symmetric three matrix model (13), plus the multitrace corrections induced by the integral over  $m$ , is a matrix model describing a non-commutative gauge theory on the fuzzy sphere. The multitrace corrections are all subleading in  $N$ , with the exception of the term  $(\text{Tr}X_a^2)^2$  in the model (13), and they depend generically on the mass deformations  $\text{Tr}X_a^2$  and  $i\epsilon_{abc} \text{Tr}X_a X_b X_c$ .

The model (13) with  $D = 0$  has been shown in [22] to sustain an absolutely stable emergent fuzzy sphere with fluctuation given by a non-commutative  $U(1)$  gauge theory very weakly coupled to a scalar field (playing the role of dark energy). In the full multitrace model there exists always the possibility of a phase transition from the fuzzy two-sphere phase to a Yang-Mills matrix phase [25] which occurs at some finite value of the gauge coupling constant. However, this is not expected since  $(\text{Tr}X_a^2)^2$  depends only on the zero mode of the normal scalar field while all other multitrace corrections are subleading in  $N$ . In other words, the full multitrace matrix model is also expected to sustain an absolutely stable emergent fuzzy sphere with fluctuation given by a slightly generalized non-commutative  $U(1)$  gauge theory coupled to a scalar field.

In summary, the multitrace matrix model (2) gives a novel and a much simpler alternative to Connes noncommutative geometry [20] and to the IKKT matrix model [21] for emergent geometry in two dimensions.

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- [1] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, Commun. Math. Phys. **59**, 35 (1978).
  - [2] Y. Shimamune, Phys. Lett. B **108**, 407 (1982).
  - [3] D. O'Connor and C. Saemann, JHEP **0708**, 066 (2007).
  - [4] C. Saemann, JHEP **1504**, 044 (2015).
  - [5] S. Rea and C. Saemann, JHEP **1511**, 115 (2015).
  - [6] J. Hoppe, MIT Ph.D. Thesis, (1982).
  - [7] J. Madore, Class. Quant. Grav. **9**, 69 (1992).
  - [8] F. Lizzi, P. Vitale and A. Zampini, JHEP **0308**, 057 (2003).
  - [9] L. Onsager, Phys. Rev. **65**, 117 (1944).
  - [10] R. C. Myers, JHEP **9912**, 022 (1999).
  - [11] T. Azuma, S. Bal, K. Nagao and J. Nishimura, JHEP **0405**, 005 (2004).
  - [12] A. Y. Alekseev, A. Recknagel and V. Schomerus, JHEP **0005**, 010 (2000).
  - [13] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, Nucl. Phys. B **604**, 121 (2001).
  - [14] U. Carow-Watamura and S. Watamura, Commun. Math. Phys. **212**, 395 (2000).
  - [15] H. Steinacker, JHEP **0503**, 075 (2005).
  - [16] V. P. Nair, A. P. Polychronakos and J. Tekel, Phys. Rev. D **85**, 045021 (2012).
  - [17] A. P. Polychronakos, Phys. Rev. D **88**, 065010 (2013).
  - [18] J. Tekel, Phys. Rev. D **87**, no. 8, 085015 (2013).
  - [19] H. Steinacker, Nucl. Phys. B **679**, 66 (2004).
  - [20] A. Connes, Commun. Math. Phys. **182**, 155 (1996).
  - [21] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B **498**, 467 (1997).
  - [22] B. Ydri, R. Ahlam and R. Khaled, arXiv:1607.08296 [hep-th].
  - [23] B. Ydri, A. Rouag and K. Ramda, Phys. Rev. D **93**, no. 6, 065055 (2016).
  - [24] B. Ydri, K. Ramda and A. Rouag, Phys. Rev. D **93**, no. 6, 065056 (2016).
  - [25] In Yang-Mills matrix models with a non-zero Chern-Simons term there exists typically a phase transition from the fuzzy sphere phase to a Yang-Mills matrix phase.